J. Differential Equations 247 (2009) 530–542

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The stability of the equilibrium of a nonlinear planar system and application to the relativistic oscillator

Jifeng Chu [a,*], Jinzhi Lei [b], Meirong Zhang [c]

[a] *Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China*
[b] *Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing 100084, China*
[c] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*

A R T I C L E I N F O

Article history:
Received 26 October 2008
Available online 10 December 2008

MSC:
34C25
34D20

Keywords:
Lyapunov stability
Periodic solutions
Planar nonlinear systems
Birkhoff normal forms
Twist coefficient
Relativistic oscillator

A B S T R A C T

In this paper we present a sufficient condition for the stability of the equilibrium of a nonlinear planar system. The proof is based on the computation of the corresponding Birkhoff normal forms. The result does not involve small parameters. Applications to the relativistic oscillator are also given.

1. Introduction

This paper is motivated by the study of the Lyapunov stability of periodic solutions of nonlinear time-periodic planar systems, which is an important problem in the theory of dynamical systems. Classical tools such as the Birkhoff normal forms and the Moser twist theorem [12,20] can yield some answer to the stability in a qualitative way. Based on these theoretical tools, an analytical method, called the third order approximation, for studying time-periodic Lagrangian equations has been developed recently by Ortega in a series of papers [15–18]. In order to compute the twist coefficients of elliptic solutions of Lagrangian equations, Ortega [17] has ingeniously established a result on the Hill

* Corresponding author.
E-mail addresses: chujf05@mails.tsinghua.edu.cn (J. Chu), jzlei@mail.tsinghua.edu.cn (J. Lei), mzhang@math.tsinghua.edu.cn (M. Zhang).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.11.013

equation

$$\ddot{x} + a(t)x = 0. \tag{1.1}$$

That is, when (1.1) is elliptic, there exists a translation of time

$$s = \alpha(t - t_0), \quad t_0 \in \mathbb{R}, \ \alpha > 0, \tag{1.2}$$

which transforms (1.1) into an R-elliptic one, i.e., the monodromy matrix of the transformed equation is a rigid rotation. This result has some independent interest, because it has given the more precise information that the monodromy matrix is a planar symplectic matrix. A nice illustrating example for Ortega's approach is the so-called swing [13,18]

$$\ddot{x} + a(t)\sin x = 0, \quad a(t) > 0, \ a(t) \in C(\mathbb{R}/T\mathbb{Z}). \tag{1.3}$$

That is, the equilibrium $x(t) = 0$ of the swing (1.3) is stable if and only if its cubic approximation

$$x'' + a(t)x - \frac{a(t)}{6}x^3 = 0$$

is stable.

After that, some researchers have extended the applications of the third order approximation, and some important stability results for several types of Lagrangian equations have been established. We refer the reader to [6,13] for the forced pendulum, and [3,23,24] for the singular equations.

However, the corresponding analytical method for studying the stability for general nonlinear planar systems has not been established in the literature up to now. The goal of this paper is to fill this gap and to bring the reader's attention to this topic. To do this, we first establish one basic result on a certain class of linear planar Hamiltonian systems in Section 2. That is, when the linear periodic planar Hamiltonian system is elliptic, there always exists a translation of time, which transforms the systems into an R-elliptic one.

Based on this fact for linear systems, it is possible to follow the works of Ortega to develop some further analytical method for studying the Lyapunov stability for more general nonlinear planar systems, at least when the linearization systems can be reduced to Hill equations. However, since planar nonlinear systems are more complicated than scalar Lagrangian equations, as an initial step toward the general method, we will consider, in Section 3, the following nonlinear system

$$\begin{cases} \dot{x} = a(t)y + c(t)y^{2n-1} + \dfrac{\partial G}{\partial y}(t, x, y), \\ \dot{y} = -b(t)x - d(t)x^{2n-1} - \dfrac{\partial G}{\partial x}(t, x, y), \end{cases} \tag{1.4}$$

where a, b, c, d are T-periodic functions and $n \geqslant 2$, and $G : \mathbb{R} \times B_\epsilon(0) \to \mathbb{R}$ is a continuous function with continuous derivatives of all orders with respect to the second variable, T-periodic in t and

$$G(t, x, y) = O\left(\left(x^2 + y^2\right)^{n+1/2}\right), \quad (x, y) \to (0, 0), \tag{1.5}$$

uniformly with respect to $t \in \mathbb{R}$. For such an example, a quite complete Lyapunov stability result for the trivial solution will be proved.

The proof of the main theorem of this paper is based on a careful computation of certain Birkhoff normal forms together with some stability results of fixed points of area-preserving maps in the plane, which were proved by Ortega in [16], and the proof was based on application of the twist theorem.

As an application, in Section 4, we apply the stability result obtained in Section 3 to a relevant physical example: the relativistic oscillator. More generally, we consider the stability of the equilibrium of a Φ-Laplacian equation

$$\big(\Phi(x')\big)' = f(t,x), \tag{1.6}$$

where Φ is a suitable increasing homeomorphism with $\Phi(0) = 0$. From a dynamical perspective, the most relevant example is the relativistic operator

$$\Phi(s) = \frac{s}{\sqrt{1-s^2}}.$$

In this case, the equation can be seen as the model for the motion of a particle moving on a straight line subjected to an external field and to relativistic effects [4]. For general Φ-Laplacian equations, some existence results for (1.6) have been proved by Mawhin and his co-workers in [1,2,10,11], where tools like Leray–Schauder degree, Mawhin continuation theorem, the method of upper and lower solutions are used. However, as far as we know, the stability of the periodic solutions has not been considered before, and only the recent work [14] can be cited.

2. Reduction from ellipticity to R-ellipticity

In this section, we will prove some basic results on linear planar Hamiltonian systems. Consider the nonlinear planar Hamiltonian system

$$\dot{y} = J\nabla_y H(t,y), \tag{2.1}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the symplectic matrix, and $H = H(t,y) : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a smooth Hamiltonian.

Suppose that $H(t,y)$ is T-periodic in t and (2.1) has a T-periodic solution $y = \varphi(t)$. By the change of variables $y = \varphi(t) + \tilde{y}$, (2.1) can be rewritten as

$$\dot{\tilde{y}} = JB(t)\tilde{y} + \nabla_{\tilde{y}}\tilde{H}(t,\tilde{y}), \tag{2.2}$$

where $B(t) = \nabla_y H(t,\varphi(t))$ and $\tilde{H}(t,y)$ is a Hamiltonian with $\tilde{H}(t,0) = 0$ and $\nabla_{\tilde{y}}\tilde{H}(t,0) = 0$. Write $\tilde{y} = (\tilde{y}_1,\tilde{y}_2)^T$. The linear part of (2.2) is a linear Hamiltonian system

$$\dot{\tilde{y}} = JB(t)\tilde{y}, \tag{2.3}$$

where

$$B(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix},$$

with $\alpha(t)$, $\beta(t)$, $\gamma(t)$ being smooth T-periodic functions.

Lemma 2.1. *(See [7].) There exists a smooth function $t \to \psi(t)$ such that the change of variables*

$$\tilde{y} = R_{\psi(t)}x \tag{2.4}$$

will transform (2.3) *into a simpler linear Hamiltonian system*

$$\dot{x}_1 = a(t)x_2, \qquad \dot{x}_2 = -b(t)x_1. \tag{2.5}$$

Here, for $\theta \in \mathbb{R}$, $R_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ *is the rigid rotation.*

The function $\psi(t)$ was constructed explicitly in [7]. In case $B(t)$ is T-periodic, by checking the construction there, one can see that $R_{\psi(t)}$ is also T-periodic. Since the transformation (2.4) is canonical, it will transform (2.2) into a Hamiltonian system as well. After these spatial changes, without loss of generality, we only consider linear systems of form (2.5).

For the linear system (2.5), the Poincaré matrix, or the Poincaré map, is

$$M = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \psi_1(T) & \psi_2(T) \end{pmatrix},$$

where $(\phi_1(t), \psi_1(t))^T$ and $(\phi_2(t), \psi_2(t))^T$ are real-valued solutions of (2.5) satisfying $\phi_1(0) = 1$, $\psi_1(0) = 0$ and $\phi_2(0) = 0$, $\psi_2(0) = 1$, respectively. The eigenvalues $\lambda_{1,2}$ of M are called the Floquet multipliers of (2.5). Obviously $\lambda_1 \cdot \lambda_2 = 1$. We can distinguish (2.5) in the following three cases:

- *elliptic*: $\lambda_1 = \bar{\lambda}_2$, $|\lambda_1| = 1$, $\lambda_1 \neq \pm 1$;
- *hyperbolic*: $0 < |\lambda_1| < 1 < |\lambda_2|$;
- *parabolic*: $\lambda_1 = \lambda_2 = \pm 1$.

It is well known that (2.5) is stable in the sense of Lyapunov if and only if (2.5) is elliptic, or is parabolic with further property that all solutions of (2.5) are T-periodic solutions in case $\lambda_1 = \lambda_2 = 1$, or $2T$-periodic solutions in case $\lambda_1 = \lambda_2 = -1$. See, for example, [5, Theorem 7.2].

For convenience, we write the linear system (2.5) into the equivalent form

$$\dot{X} = A(t)X \tag{2.6}$$

with

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & a(t) \\ -b(t) & 0 \end{pmatrix}.$$

In order to emphasize the dependence of M upon the matrix A and the period T, we write the Poincaré matrices as

$$M = M(A, T).$$

In the following we are concerned with the elliptic case, where the Floquet multipliers of (2.6) are λ and $\bar{\lambda}$ with $\lambda = \exp(i\theta)$, $\theta > 0$, $\theta \neq n\pi$, $n = 1, 2, \ldots$. In general, $M(A, T)$ is conjugate, in the symplectic group

$$\mathrm{Symp}(\mathbb{R}^2) = \{M : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is linear, and } \det M = 1\},$$

to the rigid rotation R_θ. Note that $\mathrm{Symp}(\mathbb{R}^2)$ is a 3-dimensional Lie group.

Definition 2.2. We say that the system (2.6) is *R-elliptic*, if (2.6) is elliptic and its Poincaré matrix is a rigid rotation.

Our task is to use some temporal–spatial changes to reduce (2.6) into a simpler one. To this end, let us introduce the fundamental matrix solution $\Phi(t) = \Phi(t, A)$ of (2.6), which is the matrix solution of (2.6) satisfying $\Phi(0) = \mathrm{I}_2$. Evidently, $M(A, T) = \Phi(T, A)$.

At first we consider temporal changes in (1.2)

$$T_{\alpha,t_0}(t) := \alpha(t - t_0), \quad t \in \mathbb{R}, \tag{2.7}$$

where $\alpha > 0$, $t_0 \in \mathbb{R}$. These form a Lie group of dimension 2. Under the changes $s = T_{\alpha,t_0}(t)$ and $Y(s) = X(t_0 + s/\alpha)$, system (2.6) is transformed into a new Hamiltonian system

$$Y'(s) = A_{\alpha,t_0}(s)Y(s), \tag{2.8}$$

where $A_{\alpha,t_0}(s) = \alpha^{-1}A(t_0 + s/\alpha)$ is $T_\alpha := \alpha T$ periodic in s and $' = \frac{\mathrm{d}}{\mathrm{d}s}$. The fundamental matrix solution of (2.8) is given by

$$\Phi(s, A_{\alpha,t_0}) \equiv \Phi(t_0 + s/\alpha, A)\Phi(t_0, A)^{-1}.$$

Hence, the corresponding Poincaré matrix $M(A_{\alpha,t_0}, T_\alpha)$ of (2.8) is given by

$$\begin{aligned} M(A_{\alpha,t_0}, T_\alpha) &= \Phi(t_0 + T, A)\Phi(t_0, A)^{-1} \\ &= \Phi(t_0, A)\Phi(T, A)\Phi(t_0, A)^{-1} \\ &= \Phi(t_0, A)M(A, T)\Phi(t_0, A)^{-1}. \end{aligned} \tag{2.9}$$

Here the equality $\Phi(t_0+T, A) = \Phi(t_0, A)\Phi(T, A)$ follows from the T-periodicity of $A(t)$. Equality (2.9) shows that the temporal change (2.7) will yield a spatial conjugacy (in the group $\mathrm{Symp}(\mathbb{R}^2)$) between the Poincaré maps $M(A, T)$ and $M(A_{\alpha,t_0}, T_\alpha)$.

Lemma 2.3. *Assume that* (2.6) *is elliptic with the Floquet multipliers* λ *and* $\bar{\lambda}$ *with* $\lambda = \exp(i\theta)$, $\theta > 0$, $\theta \neq n\pi$, $n = 1, 2, \ldots$. *Moreover, assume that* $a(t)$ *in* (2.6) *satisfies*

$$a(t) \neq 0 \quad \textit{for all } t \in \mathbb{R}. \tag{2.10}$$

Then there exist $t_0 \in \mathbb{R}$ *and* $\alpha > 0$ *such that*, *by taking* $D_\alpha = \mathrm{diag}(\alpha, \alpha^{-1})$, *one has either*

(i) $M(A_{\alpha,t_0}, T_\alpha) = D_\alpha R_\theta D_\alpha^{-1}$, *or*
(ii) $M(A_{\alpha,t_0}, T_\alpha) = D_\alpha R_{-\theta} D_\alpha^{-1}$.

Proof. Let $\lambda = \exp(i\theta)$ be a Floquet multiplier of (2.6). Take any (complex) eigenvector $v \in \mathbb{C}^2$ of $M(A, T)$ corresponding to the eigenvalue λ. Then the complex-valued solution $(x_1(t), x_2(t))^T$ of (2.6) with initial condition $X(0) = v$ satisfies

$$x_1(t + T) = \lambda x_1(t) \quad \text{and} \quad x_2(t + T) = \lambda x_2(t) \quad \text{for all } t \in \mathbb{R}. \tag{2.11}$$

As $|\lambda| = 1$, the function $t \to |x_1(t)|^2$ is real-valued and T-periodic. Hence there exists $t_0 \in \mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|x_1(t)\right|^2 = 0 \quad \text{at } t = t_0.$$

Since $x_1(t)$ is nontrivial, we can choose t_0 such that $x_1(t_0) \neq 0$. Define

$$\phi(t) = x_1(t)/x_1(t_0), \qquad \psi(t) = x_2(t)/x_1(t_0).$$

Then we have

$$\phi(t_0)=1,\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big|\phi(t)\big|^2=0\quad \text{at } t=t_0. \tag{2.12}$$

Set $\phi(t)=\phi_1(t)+i\phi_2(t)$ and $\psi(t)=\psi_1(t)+i\psi_2(t)$. The equality (2.12) shows that

$$\phi_1(t_0)=1,\qquad \phi_2(t_0)=0,\qquad \phi_1'(t_0)=0. \tag{2.13}$$

From the first equation of (2.5), we have from (2.13)

$$\psi(t_0)=\dot\phi(t_0)/a(t_0)=i\dot\phi_2(t_0)/a(t_0)$$

is purely imaginary. Hence

$$\psi_1(t_0)=0,\qquad \psi_2(t_0)\neq 0, \tag{2.14}$$

because $\psi(t)$ is nontrivial.

Since $(\phi(t),\psi(t))^T$ is a complex-valued solution of (2.6), we know that $(\phi_k(t),\psi_k(t))^T$, $k=1,2$, are real-valued solutions of (2.6). Hence

$$X_1(t):=\big(\phi_1(t),\psi_1(t)\big)^T,\qquad X_2(t):=\big(\phi_2(t)/\psi_2(t_0),\psi_2(t)/\psi_2(t_0)\big)^T$$

are again two real-valued solutions of (2.6). By conditions (2.13) and (2.14), one sees that

$$X_1(t_0)=(1,0)^T,\qquad X_2(t_0)=(0,1)^T.$$

Hence they are two linearly independent real-valued solutions of (2.6). We conclude that the fundamental matrix solution of (2.6) satisfies

$$\Phi(t,A)\Phi(t_0,A)^{-1}=\big(X_1(t),X_2(t)\big).$$

In particular,

$$\begin{aligned} M(A_{\alpha,t_0},T_\alpha)&=\Phi(t_0+T,A)\Phi(t_0,A)^{-1}\\ &=\begin{pmatrix}\phi_1(t_0+T) & \phi_2(t_0+T)/\psi_2(t_0)\\ \psi_1(t_0+T) & \psi_2(t_0+T)/\psi_2(t_0)\end{pmatrix}=:\tilde M.\end{aligned}$$

Denote $\lambda=\nu_1+i\nu_2$, where $\nu_k\in\mathbb{R}$, $k=1,2$. By the facts (2.11) and (2.13), (2.14), we have

$$\phi_1(t_0+T)=\nu_1,\qquad \phi_2(t_0+T)=\nu_2,$$

$$\psi_1(t_0+T)=-\nu_2\psi_2(t_0),\qquad \psi_2(t_0+T)=\nu_1\psi_2(t_0).$$

Therefore,

$$\tilde M=\begin{pmatrix}\nu_1 & \nu_2/\psi_2(t_0)\\ -\nu_2\psi_2(t_0) & \nu_1\end{pmatrix}.$$

It is easy to verify that $(1,i\psi_2(t_0))^T$ is an eigenvector of $\tilde M$ with the eigenvalue λ. By an elementary fact from linear algebra, see Lemma 2.4 below, we know that

$$\tilde M=PR_\theta P^{-1},\quad \text{where } P=\operatorname{diag}\big(1,-\psi_2(t_0)\big). \tag{2.15}$$

Take $\alpha = |\psi_2(t_0)|^{-1/2} > 0$. In case that $\psi_2(t_0) < 0$, (2.15) can be rewritten as $\tilde{M} = D_\alpha R_\theta D_\alpha^{-1}$, and in case $\psi_2(t_0) > 0$, (2.15) is $\tilde{M} = D_\alpha R_{-\theta} D_\alpha^{-1}$. These lead to case (i) and case (ii), respectively, in the statement of the lemma. □

The fact on linear algebra used in the above proof reads as follows.

Lemma 2.4. *(See [16, Lemma 4].) Let Q be a real 2×2 matrix with eigenvalues λ and $\bar{\lambda}$, an eigenvector $\omega = (a + ib, c + id)^T \in \mathbb{C}^2$ associated with λ. Set*

$$P = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}.$$

Then $M = P R_\theta P^{-1}$.

Lemma 2.3 asserts that, after temporal changes (2.7), the Poincaré matrices of (2.6) are conjugated to rotations via conjugacies in a sub-class of the group $\mathrm{Symp}(\mathbb{R}^2)$. The sub-class consists of the diagonal matrices D_α, $\alpha > 0$. However, corresponding to one λ, there are two choices of the rotations.

Now we consider, for linear Hamiltonian systems (2.6), the temporal–spatial changes of the form

$$s = \alpha(t - t_0), \qquad x^*(s) = x(t_0 + s/\alpha), \qquad y^*(s) = \alpha^{-1} y(t_0 + s/\alpha), \tag{2.16}$$

where $t_0 \in \mathbb{R}$, $\alpha > 0$. Then the linear system (2.6) is transformed into

$$\dot{x}^* = a^*(s) y^*, \qquad \dot{y}^* = -b^*(s) x^*, \tag{2.17}$$

where $a^*(s) = a(t_0 + s/\alpha)$ and $b^*(s) = \alpha^{-2} b(t_0 + s/\alpha)$ are now $T^* = \alpha T$ periodic in s. Note that, in the changes (2.16), one has different scales for x and y.

The following theorem is the main result of this section.

Theorem 2.5. *Assume that* (2.6) *is as in Lemma* 2.3. *Then there always exist some $\alpha > 0$ and $t_0 \in \mathbb{R}$ such that under the transformation* (2.16), *the transformed Poincaré matrix is a rigid rotation R_θ.*

Proof. Let t_0, α be as in Lemma 2.3. We have

$$\Phi(t_0 + T, t_0) = D_\alpha R_\theta D_\alpha^{-1}, \tag{2.18}$$

where $\exp(i\theta)$, $\theta \in \mathbb{R}$, is one of the Floquet multipliers of (2.6). Obviously, (2.17) is periodic with period $T^* = \alpha T$ after the transformation (2.16). Suppose that $(\phi_1(t, t_0), \psi_1(t, t_0))^T$ and $(\phi_2(t, t_0), \psi_2(t, t_0))^T$ are linearly independent real-valued solutions of (2.5) with $\phi_1(t_0) = \psi_2(t_0) = 1$ and $\phi_2(t_0) = \psi_1(t_0) = 0$, respectively. One may easily verify that $(\phi_1^*(\tau, 0), \psi_1^*(\tau, 0))^T$ and $(\phi_2^*(\tau, 0), \psi_2^*(\tau, 0))^T$ are linearly independent real solutions of (2.17) satisfying $\phi_1^*(0) = \psi_2^*(0) = 1$ and $\phi_2^*(0) = \psi_1^*(0) = 0$, respectively. Now (2.18) implies that the monodromy matrix corresponding to the transformed system (2.17) satisfies $\Phi^*(T^*, 0) = R_\theta$. □

3. Main results

Since the proof of the main results is based on the theory of stability of fixed points of area-preserving maps in the plane, we recall some basic results in this field. These results were proved by Ortega in [16] and some similar results can be found in [13,21].

Let $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ be an area-preserving map defined in an open neighborhood of the origin, $z = 0$ is a fixed point of F. It is assumed further that F is sufficiently smooth. For convenience, the complex notation $F = F(z, \bar{z})$ is used.

Lemma 3.1. *(See [16, Lemma 3.1].) Assume that for some $m \geqslant 3$,*

$$F(z,\bar{z}) = \lambda z + O\big(|z|^{m-1}\big), \quad z \to 0 \ \big(\lambda \in S^1\big).$$

Then there exists $H = H(z,\bar{z})$, a real-valued homogeneous polynomial of degree m such that

$$F(z,\bar{z}) = \lambda\big(z + 2i\partial_{\bar{z}} H(z,\bar{z}) + O\big(|z|^m\big)\big), \quad z \to 0. \tag{3.1}$$

Now we assume that F satisfies the conditions of Lemma 3.1 with $m = 2n$, $n \geqslant 2$. The polynomial H given by the lemma can be expressed in the form

$$H(z,\bar{z}) = \beta|z|^{2n} + \sum_{k=0}^{n-1}\big(\alpha_k z^k \bar{z}^{2n-k} + \bar{\alpha}_k \bar{z}^k z^{2n-k}\big), \tag{3.2}$$

where $\beta \in \mathbb{R}$ and $\alpha_k \in \mathbb{C}$, $k = 0, \dots, n-1$.

Lemma 3.2. *(See [16, Proposition 3.2].) Assume that $\lambda \in S^1$, F satisfies* (3.1) *with $m = 2n$ for some $n \geqslant 2$, H is given by* (3.2) *and one of the following conditions holds*:

(C$_1$) *$\lambda^{2p} \neq 1$ for each $p = 1, \dots, n$ and $\beta \neq 0$;*
(C$_2$) *$\lambda^{2p} = 1$ for some $p = 1, \dots, n$ and $H^{\#}(z,\bar{z}) \neq 0$ for each $z \in \mathbb{C} - \{0\}$, where*

$$H^{\#}(z,\bar{z}) = \frac{1}{2p}\sum_{r=0}^{2p-1} H\big(\lambda^r z, \bar{\lambda}^r \bar{z}\big).$$

Then $z = 0$ is stable with respect to F.

Now we are in position to state the main result of this paper. First we assume that (2.5) satisfies the following condition

$$\phi(t+T) = \bar{\lambda}\phi(t), \qquad \psi(t+T) = \bar{\lambda}\psi(t), \quad \forall t \in \mathbb{R}, \tag{3.3}$$

where $\lambda \in S^1$ is one of the eigenvalues $\lambda_{1,2}$ of M, and $\phi(t) = \phi_1(t) + i\phi_2(t)$, $\psi(t) = \psi_1(t) + i\psi_2(t)$. Clearly $(\phi(t), \psi(t))^T$ is the complex-valued solution of (2.5) satisfying $\phi(0) = 1$, $\psi(0) = i$.

Remark 3.3. We note that condition (3.3) is not restrictive. In fact, Lemma 2.3 guarantees that (3.3) holds for the case of ellipticity $|\lambda| = 1$, $\lambda \neq \pm 1$, after a temporal–spatial transformation. When (2.5) is parabolic and stable, condition (3.3) is always satisfied, because all solutions of (2.5) are either T-periodic or $2T$-periodic in this case. Moreover, condition (3.3) is equivalent to saying that the monodromy matrix M of (2.5) is a rotation. In fact, if $\lambda \neq \pm 1$, M is a rotation different from $\pm \mathrm{I}_2$, while M is $\pm \mathrm{I}_2$ if $\lambda = \pm 1$.

Before stating the main result, we present the following simple result on the linear inhomogeneous system

$$\begin{cases} \dot{x} = a(t)y + f(t), \\ \dot{y} = -b(t)x - g(t), \end{cases} \tag{3.4}$$

which is a consequence of the formula of variation of constants together with (3.3). Here f and g are continuous functions.

Lemma 3.4. *Let $(x(t), y(t))^T$ be the solution of* (3.4) *with* $x(0) = y(0) = 0$. *Assume that* (3.3) *is satisfied. Then we have*

$$x(T) + iy(T) = -i\lambda \int_0^T \big(f(t)\psi(t) + g(t)\phi(t)\big)\,\mathrm{d}t.$$

The main result of this section reads as follows.

Theorem 3.5. *Assume that* (2.5) *is stable, and* $\int_0^T |c(t)|\,\mathrm{d}t \neq 0$, $\int_0^T |d(t)|\,\mathrm{d}t \neq 0$. *Then the trivial solution of* (1.4) *is stable, if one of the following two conditions holds*:

(H_1) $c(t) \geqslant 0, d(t) \geqslant 0$,
(H_2) $c(t) \leqslant 0, d(t) \leqslant 0$.

Proof. We prove the result assuming that condition (3.3) is satisfied. The result holds for the general case because we have Remark 3.3.

Let $(x(t), y(t))^T = (x(t, z, \bar{z}), y(t, z, \bar{z}))^T$ be the solution of the nonlinear system (1.4) with $x(0) = q$ and $y(0) = p$, where $z = q + ip$. The theorem of differentiability with respect to initial conditions implies that

$$x(t, z, \bar{z}) = \frac{\bar{\phi}(t)z + \phi(t)\bar{z}}{2} + O\big(|z|^2\big), \quad z \to 0, \tag{3.5}$$

$$y(t, z, \bar{z}) = \frac{\bar{\psi}(t)z + \psi(t)\bar{z}}{2} + O\big(|z|^2\big), \quad z \to 0. \tag{3.6}$$

These two expansions are uniform in $t \in [0, T]$.

We look at the nonlinear system (1.4) as one system of the kind (3.4) with

$$f(t) = c(t)y^{2n-1}(t) + \frac{\partial G}{\partial y}\big(t, x(t), y(t)\big), \tag{3.7}$$

$$g(t) = d(t)x^{2n-1}(t) + \frac{\partial G}{\partial x}\big(t, x(t), y(t)\big). \tag{3.8}$$

Note from (1.5) and (3.5), (3.6) that

$$\frac{\partial G}{\partial x}\big(t, x(t), y(t)\big),\ \frac{\partial G}{\partial y}\big(t, x(t), y(t)\big) = O\big(|z|^{2n}\big) \quad \text{as } |z| \to 0.$$

Since we have assumed that (3.3) holds, we can apply Lemma 3.4 to (3.4) to obtain

$$P(z, \bar{z}) = \lambda z - i\lambda \int_0^T g(t)\phi(t)\,\mathrm{d}t - i\lambda \int_0^T f(t)\psi(t)\,\mathrm{d}t, \tag{3.9}$$

where f, g are given by (3.7) and (3.8).

Combining (3.5), (3.6) and (3.9), we have the expansion

$$\begin{aligned} P(z, \bar{z}) = \lambda z &- i\lambda \int_0^T c(t)\phi(t)\left(\frac{\bar{\phi}(t)z + \phi(t)\bar{z}}{2}\right)^{2n-1} \mathrm{d}t \\ &- i\lambda \int_0^T d(t)\psi(t)\left(\frac{\bar{\psi}(t)z + \psi(t)\bar{z}}{2}\right)^{2n-1} \mathrm{d}t + O\big(|z|^{2n}\big). \end{aligned}$$

Then P satisfies (3.1) with H given by

$$H(z,\bar z) = -\frac{1}{2^{2n}}\int_0^T \frac{c(t)(\bar\phi(t)z+\phi(t)\bar z)^{2n}}{2n}\,\mathrm{d}t - \frac{1}{2^{2n}}\int_0^T \frac{d(t)(\bar\psi(t)z+\psi(t)\bar z)^{2n}}{2n}\,\mathrm{d}t.$$

The coefficient β in (3.2) is given by

$$\beta = -\frac{1}{2^{2n+1}n}\binom{2n}{n}\left(\int_0^T c(t)\left|\phi(t)\right|^{2n}\mathrm{d}t + \int_0^T d(t)\left|\psi(t)\right|^{2n}\mathrm{d}t\right).$$

Since $(\phi_1(t),\psi_1(t))^T$ and $(\phi_2(t),\psi_2(t))^T$ are linearly independent solutions of (2.5), we know $|\phi(t)| \neq 0$, $|\psi(t)| \neq 0$ for all $t \in \mathbb{R}$.

Assume that condition (H_1) holds. Then $\beta < 0$ and $H(z,\bar z) < 0$ for all $z \in \mathbb{C} - \{0\}$. When $\lambda^{2p} \neq 1$ for each $p = 1,\dots,n$, (C_1) is satisfied. If $\lambda^{2p} = 1$ for some $p = 1,\dots,n$, then the definition of $H^{\#}$ and the negativity of H imply that $H^{\#}(z,\bar z) < 0$ for all $z \in \mathbb{C} - \{0\}$, and therefore (C_2) holds. If condition (H_2) is satisfied, the result holds by similar analysis. □

Remark 3.6. In Theorem 3.5, we have reduced the Lyapunov stability for the equilibrium $(x,y) = (0,0)$ of nonlinear system (1.4) to that of the corresponding linear system (2.5). For linear systems (2.5), the stability problem can be studied using eigenvalue theory. Some stability criteria are established in [19,25]. For example, if $a(t) > 0$ and $b(t) > 0$ satisfy

$$\left(\int_0^T a(t)\,\mathrm{d}t\right)\cdot\left(\int_0^T b(t)\,\mathrm{d}t\right) \leqslant 4, \tag{3.10}$$

then (2.5) is in the first elliptic region and is therefore Lyapunov stable. Moreover, the constant 4 in this condition is optimal.

Every stability criterion for the linear system together with the assumption (H_1) or (H_2) produces a stability criterion for the nonlinear system (1.4). For example, we have

Corollary 3.7. *Assume that $a(t) > 0$, $b(t) > 0$ satisfying* (3.10), $\int_0^T |c(t)|\,\mathrm{d}t \neq 0$, $\int_0^T |d(t)|\,\mathrm{d}t \neq 0$. *Then the trivial solution of* (1.4) *is stable if* (H_1) *or* (H_2) *is satisfied.*

As an example, we apply Theorem 3.5 to the following planar system

$$\begin{cases} \dot x = h(t)\sin y, \\ \dot y = -l(t)\sin x. \end{cases} \tag{3.11}$$

Corollary 3.8. *Assume that $h(t)$, $l(t)$ are positive T-periodic functions. Then the trivial solution of* (3.11) *is stable if the linearized system is stable.*

Proof. System (3.11) can be regarded as one of (1.4) with $n = 2$ and

$$a(t) = h(t), \qquad b(t) = l(t), \qquad c(t) = -\frac{h(t)}{6}, \qquad d(t) = -\frac{l(t)}{6}.$$

Since $h(t)$, $l(t)$ are positive T-periodic functions, the result is now a direct consequence of Theorem 3.5. $\square$

Remark 3.9. In [9], Liu studied the stability of the trivial solution of the system

$$\begin{cases} \dot{x} = c(t)y^{2m+1} + f_1(x, y, t), \\ \dot{y} = -d(t)x^{2n+1} - f_2(x, y, t) \end{cases} \tag{3.12}$$

with $m, n \in \mathbb{Z}_+$, $m+n \geqslant 1$. It was required that $c(t)$, $d(t)$ are even functions and f_1, f_2 are real analytic in a neighborhood of the origin. Moreover, it was assumed that (3.12) is reversible with respect to the involution $G : (x, y) \to (-x, y)$. Note that, under these assumptions, the equilibrium $(x, y) = (0, 0)$ of (3.12) is always parabolic. However, in Theorem 3.5, both the elliptic and (stable) parabolic cases are allowed. Therefore, in some sense, our results improve and generalize those in [9] for $m = n$.

4. Application to the relativistic oscillator

In this section, we study the Lyapunov stability of the equilibrium of (1.6), using the formula given in Section 3. Throughout this section, we always assume that $\Phi : (-a, a) \to (-b, b)$ is an increasing odd homeomorphism with $\Phi(0) = 0$ and $0 < a, b \leqslant \infty$, such that $\Phi^{-1} \in C^3$ can be expanded around 0 as

$$\Phi^{-1}(y) = y + ky^3 + \cdots$$

with $k = (\Phi^{-1})'''(0)/6$. Besides, it is assumed that $f(t, 0) \equiv 0$. Therefore $x = 0$ is an equilibrium of (1.6).

Usually, a Φ-Laplacian operator is said *singular* when the domain of Φ is finite (that is, $a < +\infty$), on the contrary the operator is said *regular*. On the other hand we say that Φ is *bounded* if its range is finite (that is, $b < +\infty$) and *unbounded* in other case. There are three paradigmatic models in this context:

- $a = b = +\infty$ (regular unbounded): the p-Laplacian operator

$$\Phi_1(x) = |x|^{p-2}x \quad \text{with } p > 1.$$

- $a < +\infty$, $b = +\infty$ (singular unbounded): the relativistic operator

$$\Phi_2(x) = \frac{x}{\sqrt{1 - x^2}}.$$

- $a = +\infty$, $b < +\infty$ (regular bounded): the one-dimensional mean curvature operator

$$\Phi_3(x) = \frac{x}{\sqrt{1 + x^2}}.$$

Our results will cover the latter two examples, but as it was noted in the Introduction, from a dynamical point of view the relativistic operator is perhaps the more relevant case.

Let $\Phi_2(x') = y$. Then (1.6) can be written in the equivalent form

$$\begin{cases} \dot{x} = \Phi_2^{-1}(y), \\ \dot{y} = f(t, x). \end{cases} \tag{4.1}$$

The stability of $x=0$ of (1.6) is equivalent to the stability of the equilibrium of the planar system (4.1). To apply Theorem 3.5, we express (4.1) in the form

$$\begin{cases} \dot{x} = y + ky^3 + \cdots, \\ \dot{y} = f_x(t,0)x + \frac{1}{2} f_{xx}(t,0)x^2 + \frac{1}{6} f_{xxx}(t,0)x^3 + \cdots. \end{cases} \tag{4.2}$$

By an easy computation, $k = -\frac{1}{2}$, and therefore Theorem 3.5 can be applied.

Theorem 4.1. *Assume that $f(t,x)$ satisfies $f(t,0)=0$ for all t and the linearized equation*

$$x'' - f_x(t,0)x = 0$$

is stable. Then the equilibrium of (1.6) *is stable if the following conditions are satisfied*

(i) $f_{xx}(t,0)=0$ *for all* $t \in \mathbb{R}$;
(ii) $f_{xxx}(t,0) \geqslant 0$ *for all* $t \in \mathbb{R}$, *and* $\int_0^T |f_{xxx}(t,0)|\,\mathrm{d}t > 0$.

Using Theorem 4.1, we can study the following two important examples, and the stability results are direct from Theorem 4.1.

Example 4.2. The relativistic pendulum: The motion of a pendulum with variable length and relativistic effects can be described by

$$\big(\Phi_2(x')\big)' + \ell(t)\sin x = 0, \tag{4.3}$$

where $\ell(t)$ is a positive and continuous 2π-periodic function.

Corollary 4.3. *If the Hill equation $x'' + \ell(t)x = 0$ is stable, then the equilibrium of* (4.3) *is stable.*

Example 4.4. The relativistic Sitnikov problem: The motion of an asteroid of negligible mass in a perpendicular orbit to the plane defined by motion of two primaries in an elliptic Keplerian orbit is described by the classical Sitnikov equation [8,22]. If relativistic effects are considered, the equation is

$$\big(\Phi_2(x')\big)' + \frac{2x}{(x^2 + r(t)^2)^{3/2}} = 0, \tag{4.4}$$

where $r(t)$ is a positive and continuous 2π-periodic function, to denote the distance between the primaries.

Corollary 4.5. *If $x'' + \frac{2}{r(t)^3}x = 0$ is stable, then the equilibrium of* (4.4) *is stable.*

Acknowledgments

The research of Jifeng Chu is supported by the National Natural Science Foundation of China (Grant No. 10801044) and Jiangsu Natural Science Foundation (Grant No. BK2008356). The research of Jinzhi Lei is supported by the National Natural Science Foundation of China (Grant Nos. 10601029 and 10301006). The research of Meirong Zhang is supported by the National Basic Research Program of China (Grant No. 2006CB805903) and the National Natural Science Foundation of China (Grant Nos. 10325102 and 10531010). The authors would like to thank Professor Pedro J. Torres for his careful reading of the manuscript and valuable suggestions, especially bringing their attention to consider the relativistic oscillator.

References

[1] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular Φ-Laplacian, J. Differential Equations 243 (2007) 536–557.
[2] C. Bereanu, J. Mawhin, Periodic solutions of some nonlinear perturbations of the Φ-Laplacian with possibly bounded Φ, Nonlinear Anal. 68 (2008) 1668–1681.
[3] J. Chu, M. Zhang, Rotation numbers and Lyapunov stability of elliptic periodic solutions, Discrete Contin. Dyn. Syst. 21 (2008) 1071–1094.
[4] H. Goldstein, Classical Mechanics, second ed., Addison-Wesley, Reading, 1980.
[5] J.K. Hale, Ordinary Differential Equations, second ed., Krieger, New York, 1980.
[6] J. Lei, X. Li, P. Yan, M. Zhang, Twist character of the least amplitude periodic solution of the forced pendulum, SIAM J. Math. Anal. 35 (2003) 844–867.
[7] B.M. Levitan, L.S. Sargsjan, Sturm–Liouville and Dirac Operators, Math. Appl. (Soviet Ser.), vol. 59, Kluwer Academic, Dordrecht, 1991.
[8] J. Llibre, R. Ortega, On the families of periodic orbits of the Sitnikov problem, SIAM J. Appl. Dyn. Syst. 7 (2008) 561–576.
[9] B. Liu, The stability of the equilibrium of reversible system, Trans. Amer. Math. Soc. 351 (1999) 515–531.
[10] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations 145 (1998) 367–393.
[11] R. Manásevich, J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators, J. Korean Math. Soc. 37 (2000) 665–685.
[12] J. Moser, On invariant curves of area preserving mappings of an annulus, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1962) 1–20.
[13] D. Núñez, R. Ortega, Parabolic fixed points and stability criteria for nonlinear Hill's equation, Z. Angew. Math. Phys. 51 (2000) 890–911.
[14] D. Núñez, P.J. Torres, KAM dynamics in the driven relativistic harmonic oscillator, preprint.
[15] R. Ortega, The twist coefficient of periodic solutions of a time-dependent Newton's equation, J. Dynam. Differential Equations 4 (1992) 651–665.
[16] R. Ortega, The stability of equilibrium of a nonlinear Hill's equation, SIAM J. Math. Anal. 25 (1994) 1393–1401.
[17] R. Ortega, Periodic solution of a Newtonian equation: Stability by the third approximation, J. Differential Equations 128 (1996) 491–518.
[18] R. Ortega, The stability of the equilibrium: A search for the right approximation, in: Ten Mathematical Essays on Approximation in Analysis and Topology, Elsevier, Amsterdam, 2005, pp. 215–234.
[19] R. Ortega, M. Zhang, Optimal bounds for bifurcation values of a superlinear periodic problem, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 119–132.
[20] C. Siegel, J. Moser, Lectures on Celestial Mechanics, Springer-Verlag, Berlin, 1971.
[21] C. Simo, Stability of degenerate fixed points of analytic area preserving mappings, Astérisque 98–99 (1982) 184–194.
[22] K.A. Sitnikov, Existence of oscillating motion for the three-body problem, Dokl. Akad. Nauk 133 (1960) 303–306.
[23] P.J. Torres, Twist solutions of a Hill's equations with singular term, Adv. Nonlinear Stud. 2 (2002) 279–287.
[24] P.J. Torres, Existence and stability of periodic solutions for second order semilinear differential equations with a singular nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 196–201.
[25] M. Zhang, Sobolev inequalities and ellipticity of planar linear Hamiltonian systems, Adv. Nonlinear Stud. 8 (2008) 633–654.